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Balls in $\mathbb{R}^k$ Do Not Cut All Subsets of $k + 2$ Points[1]

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For any set E cf $k + 2$ points in $\mathbb{R}^k$, $k = 1, 2,\ldots$, not all subsets of E are of the form $B \cap E$ where B is a ball.

Let S be a set and $\mathscr{C}$ a class of subsets of S. For each finite set $F \subset S$, let $\Delta^{\mathscr{C}}(F)$ denote the number of different sets $C \cap F$, $C \in \mathscr{C}$. For $n = 1, 2,\ldots$, let

$$m^{\mathscr{C}}(n) := \max\{\Delta^{\mathscr{C}}(F) \colon \operatorname{card} F = n\}$$

where card F is the number of elements in F. Let

$$V(\mathscr{C}) := \inf\{n \colon m^{\mathscr{C}}(n) < 2^n\}, \qquad = +\infty \quad \text{if} \quad m^{\mathscr{C}}(n) = 2^n \text{ for all } n.$$

Vapnik and Červonenkis [4] introduced the above quantities and proved that $m^{\mathscr{C}}(n) \leqslant n^{V(\mathscr{C})} + 1$. In [5, p. 217], they show that $m^{\mathscr{C}}(r) \leqslant 3r^{v-1}/(2(v-1)!)$, where $v := V(\mathscr{C})$, if $r > v$. Thus either $m^{\mathscr{C}}(r) = 2^r$ for all r, or $m^{\mathscr{C}}(r)$ grows much more slowly, as a power of r. This has probabilistic implications [1, 2, 3, 4, 5].

In $\mathbb{R}^k$, let $B(k)$ denote the set of all closed balls

$$B(x, s) := \{y \colon | x - y | \leqslant s\},$$

$x \in \mathbb{R}^k$, $s > 0$, where $| \cdot |$ denotes the usual Euclidean norm. Let $\mathscr{H}(k)$ denote the set of all closed half-spaces

$$H(x, a) := \{y \colon (y, x) \geqslant a\},$$

$a \in \mathbb{R}$, $y \in \mathbb{R}^k$.

For any real-valued function g on a set X let

$$nn(g) := \{x \in X \colon g(x) \geqslant 0\}.$$

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For a set G of functions let

$$nn(G) := \{nn(g): g \in G\}.$$

The following is known ([1], [2, Theorem 7.2]):

THEOREM A. *If G is an m-dimensional vector space of real functions on a set, then $V(nn(G)) = m + 1$.*

Taking $X = \mathbb{R}^k$ and G as the space of affine functions, we obtain

THEOREM B. $V(\mathscr{H}(k)) = k + 2$ *for all* $k = 1, 2, \ldots$.

If G on $\mathbb{R}^k$ is the vector space spanned by 1, $|x|^2$, and the coordinates $x_1, \ldots, x_k$, then $B(k) \subset nn(G)$, so by Theorem A, $V(B(k)) \leqslant V(nn(G)) = 1 + \dim(G) = k + 3$. Actually, we have

THEOREM 1. $V(B(k)) = k + 2$ *for all* $k = 1, 2, \ldots$.

One might ask whether Theorem 1, like Theorem B, is implied by the general Theorem A. Suppose $B(k) \subset nn(G)$ where G is a vector space of dimension $k + 1$. Let F be a set of $k + 2$ points $v_1, \ldots, v_{k+2}$ of $\mathbb{R}^k$ where $v_1 = 0$, v_3 through v_{k+2} are linearly independent, and $v_2 = (v_3 + \cdots + v_{k+2})/(k + 1)$. Take a ball $B \in B(k)$ with center v_2 and small enough positive radius so that $F \cap B = \{v_2\}$. Take $g \in G$ with $nn(g) = B$. Take $j \in G$ with $v_2 \notin nn(j)$. Then for $h = g - \epsilon j$ and ϵ small enough, $h(v_2) > 0 > h(v_i)$ for $i \neq 2$. But then $F \backslash \{v_2\} = nn(-h) \cap F$. For any subset A of F other than these two, A and $F \backslash A$ have disjoint convex hulls, so they can be separated by a hyperplane, and $A = C \cap F$ for some ball C of large enough radius. Thus all subsets of F are of the form $E \cap F$, $E \in nn(G)$, contradicting Theorem A. So Theorem 1 cannot be obtained in this way.

From Theorem B and the nature of half-spaces it follows that $m^{\mathscr{H}(k)}(k + 2) \leqslant 2^{k+2} - 2$. From Theorem 1 it will follow that $m^{B(k)}(k + 2) \leqslant 2^{k+2} - 1$. Note that both these bounds are attained, for the set F in the last paragraph.

Proof of Theorem 1. Let $A \subset \mathbb{R}^k$ be any set containing $k + 2$ points $x_1, \ldots, x_{k+2}$. Let $\mathrm{co}(B)$ denote the convex hull of B. By Theorem B, and since disjoint convex sets can be separated by a hyperplane, there is some $B \subset A$ such that for $C := A \backslash B$, $\mathrm{co}(B) \cap \mathrm{co}(C) \neq \phi$. Suppose S and T are closed balls with $S \supset B$, $S \cap C = \phi$, $T \supset C$, $T \cap B = \phi$. But then $B \subset S \backslash T$ and $C \subset T \backslash S$ are strictly separated by a hyperplane, a contradiction. Q.E.D.

It would be interesting to extend Theorem 1 to suitable Riemann manifolds.

In an infinite-dimensional Hilbert space, for the class $\mathscr{B}$ of balls $V(\mathscr{B}) = +\infty$. In fact, there exists a curve C such that for every finite $F \subset C$, there is a closed ball B with $B \cap C = F$ [3, Proposition 2].

References

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